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# COMPARISON OF LINEAR AND DIMUS BEAMFORMING IN TERMS OF MISCLASSIFICATION PROBABILITIES FOR A SET OF DETERMINISTIC SIGNALS

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**ABSTRACT**

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# COMPARISON OF LINEAR AND DIMUS BEAMFORMING IN TERMS OF MISCLASSIFICATION PROBABILITIES FOR A SET OF DETERMINISTIC SIGNALS

## INTRODUCTION

An idealized communications scenario will be posed here in order to accurately determine the performance losses caused by the use of a digital multibeam steering system (DIMUS) instead of a linear signal processor during beamforming. These relative losses are of great interest in communications, especially for moderate-to-high input signal-to-noise ratios (SNRs), where saturation and distortion effects can take place in the clippers. Although the absolute levels of performance achieved by the following idealized models of linear and DIMUS beamformers (combined with crosscorrelators) may not be attainable in practice, the difference in levels should be an accurate measure of the loss incurred by the use of clipping. This difference can be investigated as a function of the input SNR, the number of channels, the number of time samples, the reference waveforms, etc., in order to pinpoint and ascertain where further effort or modified processing may be required to minimize the effects of the additional losses.

A set of M deterministic real signals $\{s_m(t)\}$ is used to communicate information to a receiver, which knows these detailed waveforms, but not which one was transmitted. If signal $m$ is transmitted, the received waveform at the k-th element of the receiver (after beamformer delay-time alignment) is

$$s_{\underline{m}}(t) + n_k(t) \quad \text{for } 1 \leq k \leq K \ , \tag{1}$$

where additive noises $\{n_k(t)\}$, $1 \leq k \leq K$, are stationary, real, zero-mean, dependent Gaussian random processes with covariances

$$E\{n_k(t)\ n_j(t-\tau)\} = R_{kj}(\tau) \quad \text{for } 1 \leq k,j \leq K \ . \tag{2}$$

The quantity $E\{\ \}$ denotes an ensemble average. Based upon the K measurements of the waveforms in equation (1) over an observation time T, a decision must be made as to which of the M signals $\{s_m(t)\}$ was actually transmitted.

The signal set $\{s_m(t)\}$ could be augmented with an additional signal, $s_o(t) = 0$, in order to include the detection scenario, where it is not known if any signal is present at all. That situation can easily be incorporated in the following developments, if desired, by a slight modification to the receiver signal processing.

## LINEAR PROCESSING

### BACKGROUND

The processor of interest in this section is depicted in figure 1. Time is normalized and discretized such that samples are taken at multiples of unit time, at instants t = 1,2,...,T.

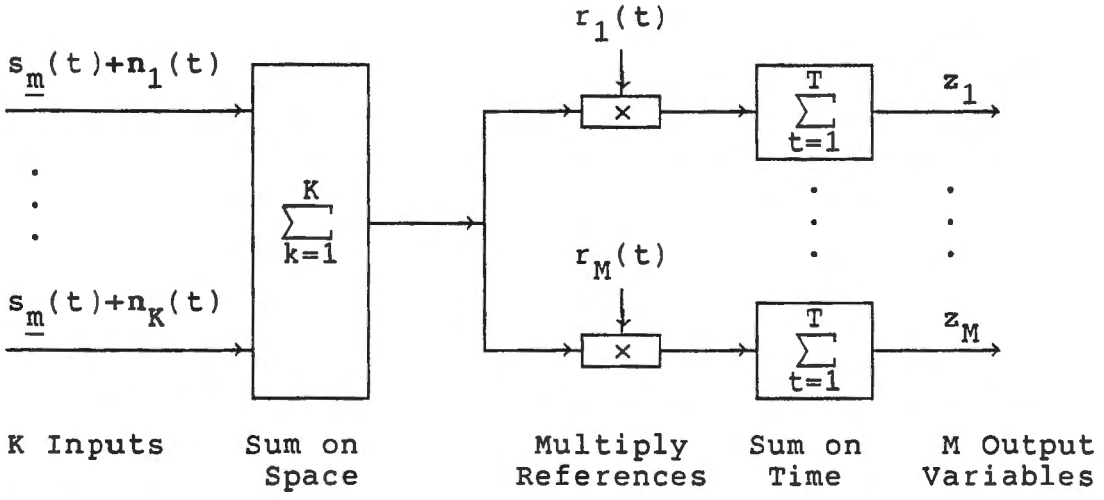


**Figure 1. Linear Processor**

The m-th local reference, $r_m(t)$, would normally be taken as some scaled version of signal $s_m(t)$, at least if the received additive noises are independent with white spectra. Generality will be allowed here by not specifying the exact choices of references $\{r_m(t)\}$. This permits reference $r_m(t)$ to be partially mismatched to the m-th signal, such as a linearly filtered version of signal $s_m(t)$; a clipped version of the signal, such as

$\text{sgn}\{s_m(t)\}$; or some other distorted version, such as $g\{s_m(t)\}$, if desired.

The output variables $Z \equiv [z_1,\ldots,z_M]^T$ in figure 1 are subject to additional processing before a decision is made as to which particular signal was transmitted. In particular, the a priori probabilities of each signal, the energies of each signal, and the costs of misclassification of each signal pair, $1 \leq \underline{m},m \leq M$, must be taken into account. The end result is that Mx1 output vector Z is transformed to the Mx1 decision vector $\underline{Z}$ according to linear transformation

$$\underline{Z} = A\ Z + B\ , \tag{3}$$

where known MxM matrix A and known Mx1 matrix B incorporate the probabilities, energies, and costs mentioned above. The signal identity decision then consists of selection of the number of the maximum variable in modified set $\underline{Z}$.

The noise inputs $\{n_k(t)\}$ in figure 1 are joint Gaussian random processes. Because all the operations on the inputs in figure 1 are linear, the output variables Z are also joint Gaussian. Therefore, the modified decision variables $\underline{Z}$ generated by equation (3) are joint Gaussian, with mean and covariance matrices given by

$$E\{\underline{Z}\} = A\ E\{Z\} + B\ , \quad \text{Cov}\{\underline{Z}\} = A\ \text{Cov}\{Z\}\ A^T\ , \tag{4}$$

respectively. This knowledge in equation (4) constitutes a complete statistical description of the decision variables $\underline{Z}$. The essential analysis problem relevant to the linear processor in figure 1 is therefore the determination of the mean and covariance matrices, namely, E{Z} and Cov{Z}, of output Z.

## SIMULATION

Once the information in equation (4) is available, the probability of correct decision for the $\underline{m}$-th transmitted signal, $P_c(\underline{m})$, requires integration of the joint Gaussian probability density function (PDF) of $\underline{Z}$ over a sector of M-dimensional space. This is an intractable analytic problem, except in the most special of cases. Accordingly, it is recommended that the M-dimensional vector $\underline{Z}$, with the known statistics (4), be simulated for purposes of estimating $P_c(\underline{m})$. Of course, this simulation must be conducted separately for each value of $\underline{m}$ in the range (1,M), since the energies, costs, and a priori probabilities will be different. However, this is not a major numerical task because the size M of the signal set is less than 10, and the correct-decision probabilities of interest are on the order of 0.9 to 0.99. This contrasts with the estimation of false alarm probabilities, which can be in the $10^{-6}$ range. The generation of random vector $\underline{Z}$ with specified statistics (4) is addressed in appendix A.

This limited simulation requirement is achieved because the linear processor in figure 1 has been analyzed exactly up through the output variables Z. Thus, it is not necessary to simulate T time samples of K channels of correlated input data waveforms, which significantly reduces the size of the simulation. Effort can be concentrated instead on a small-size M-ary decision, where the fraction of correct decisions for signal $\underline{m}$ can be accurately estimated without an excessive simulation schedule. Integer $\underline{m}$ must be varied over the range $1 \leq \underline{m} \leq M$.

## ANALYSIS

The output variables $\{z_m\}$ in figure 1 are given by linear operations on the inputs according to

$$z_m = \sum_{t=1}^{T} r_m(t) \sum_{k=1}^{K} \left[ s_{\underline{m}}(t) + n_k(t) \right] =$$

$$= K \sum_{t=1}^{T} r_m(t)\, s_{\underline{m}}(t) + \sum_{t=1}^{T} r_m(t)\, n(t) \quad \text{for } 1 \leq m \leq M \,, \qquad (5)$$

where $n(t)$ is the sum of all K received noises $\{n_k(t)\}$. The mean of output $z_m$ is given by the first term as

$$E\{z_m\} = K \sum_{t=1}^{T} r_m(t)\, s_{\underline{m}}(t) \quad \text{for } 1 \leq m \leq M \,, \qquad (6)$$

namely, K times the crosscorrelation of reference waveform m with transmitted signal $\underline{m}$.

The covariance of output random variables $z_m$ and $z_\ell$ follows from the second term in equation (5) as

$$E\{z_m \; z_\ell\} = \sum_{t,u=1}^{T} r_m(t) \; r_\ell(u) \; E\{n(t) \; n(u)\} =$$

$$= \sum_{\tau=-T}^{T} \phi_{m\ell}(\tau) \; R(\tau) \quad \text{for } 1 \le m,\ell \le M \; , \tag{7}$$

where

$$R(\tau) \equiv E\{n(t) \; n(t-\tau)\} = \sum_{k,j=1}^{K} E\{n_k(t) \; n_j(t-\tau)\} = \sum_{k,j=1}^{K} R_{kj}(\tau) \tag{8}$$

is the autocovariance of the total received stationary noise process $n(t)$, and where equations (5) and (2) have been used. The quantity $\phi_{m\ell}(\tau)$ is the autocorrelation of the references according to

$$\phi_{m\ell}(\tau) = \sum_{t} r_m(t) \; r_\ell(t-\tau) \quad \text{for } 1 \le m,\ell \le M \; ; \tag{9}$$

the sum on t is taken wherever the summand is nonzero.

Equations (6) and (7) furnish the mean and covariance information required to statistically characterize output random variables $\{z_m\}$ in figure 1. They enable the simulation described above to be carried out for any signal and reference sets. All the noise covariances in equation (2) must also be available; recall that these noises $\{n_k(t)\}$ in equation (1) and figure 1 are measured at the outputs of the steering delays in the beamformer. In terms of the measured element noises $\{\underline{n}_k(t)\}$, there follows

$n_k(t) = \underline{n}_k(t-\tau_k(\theta))$, where $\tau_k(\theta)$ is the k-th time delay required to steer in direction $\theta$.

## DIMUS PROCESSING

### BACKGROUND

In this section, the linear processor of figure 1 is modified to include clipping at the element level. The resulting DIMUS processor of interest is indicated in figure 2. The m-th local reference, $r_m(t)$, can be taken equal to $s_m(t)$; a clipped version, such as $\text{sgn}\{s_m(t)\}$; or some other distorted version, such as $g\{s_m(t)\}$, if desired.

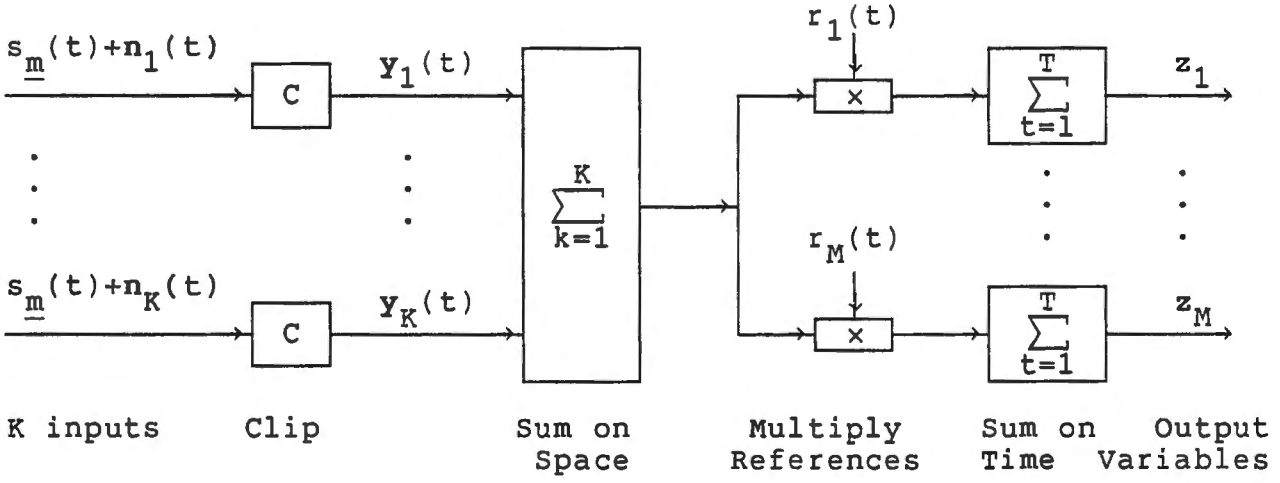


**Figure 2. DIMUS Processor**

The output variables $Z \equiv [z_1,\ldots,z_M]^T$ in figure 2 are subject to additional processing before a decision is made as to which particular signal was transmitted. Just as for the earlier

linear processor (3), output vector Z is subject to a linear transformation into decision vector $\underline{Z} = A Z + B$, where A and B are known matrices. Identity decisions are made upon $\underline{Z}$. This is not necessarily the optimum processing to conduct upon DIMUS output Z; rather, guided by the linear processor, it is adopted as a reasonable procedure to employ for the DIMUS processor as well.

When the product of the number of channels and the number of time samples in figure 2, namely, KT, is large, each output variable $z_m$ is the sum of a large number of random variables. In this case, the random output vector Z tends toward a set of joint Gaussian random variables; for a justification, see the extensive simulation using $K = 256$ and $T = 4096$ in reference 1, page 20.

The above observation allows the analytic investigation of the DIMUS processor to concentrate on finding the mean vector and the covariance matrix of output Z. Then, the corresponding mean and covariance statistics for decision vector $\underline{Z}$ are easily determined by means of equation (4), at which point a low-order M-ary simulation of classification performance can be conducted. The comments made earlier in the Simulation subsection are directly relevant here.

**ANALYSIS**

The m-th output in figure 2 is given by

$$z_m = \sum_{t=1}^{T} r_m(t) \sum_{k=1}^{K} y_k(t) \quad \text{for } 1 \le m \le M \ , \tag{10}$$

where

$$y_k(t) = \text{sgn}\{s_{\underline{m}}(t) + n_k(t)\} \quad \text{for } 1 \le k \le K \ . \tag{11}$$

Using the zero-mean Gaussian character of input noises $\{n_k(t)\}$ shows the mean of $z_m$ to be given by

$$E\{z_m\} = \sum_{t=1}^{T} r_m(t) \sum_{k=1}^{K} E\{y_k(t)\} \quad \text{for } 1 \le m \le M \ , \tag{12}$$

with

$$E\{y_k(t)\} = E\{\text{sgn}\{s_{\underline{m}}(t) + n_k(t)\}\} =$$

$$= \int dx \ \text{sgn}\{s_{\underline{m}}(t) + x\} \left(2\pi\sigma_k^2\right)^{-\frac{1}{2}} \exp\left(-\frac{x^2}{2\sigma_k^2}\right) =$$

$$= 2\,\Phi\left(s_{\underline{m}}(t)/\sigma_k\right) - 1 \quad \text{for } 1 \le k \le K \ , \tag{13}$$

where $\sigma_k^2$ is the noise variance $E\{n_k^2(t)\} = R_{kk}(0)$ in the k-th channel, and normal probability function $\Phi$ and PDF $\phi$ are given by

$$\Phi(x) \equiv \int_{-\infty}^{x} dt \ (2\pi)^{-\frac{1}{2}} \exp(-t^2/2) \equiv \int_{-\infty}^{x} dt \ \phi(t) \ . \tag{14}$$

Substitution of equation (13) into equation (12) yields mean

$$E\{z_m\} = \sum_{t=1}^{T} r_m(t) \sum_{k=1}^{K} \left[2\Phi\left(s_{\underline{m}}(t)/\sigma_k\right)-1\right] \quad \text{for } 1 \le m \le M \ . \quad (15)$$

This result requires a double sum over KT terms, just as the output $z_m$ itself does in equation (10). In the special case where all the input noise standard deviations $\sigma_k$ are equal to $\sigma$, equation (15) simplifies to

$$E\{z_m\} = K \sum_{t=1}^{T} r_m(t) \left[2\Phi\left(s_{\underline{m}}(t)/\sigma\right)-1\right] \quad \text{for } 1 \le m \le M \ . \quad (16)$$

Comparison of equation (16) with the analogous result (6) for the linear processor reveals that $s_{\underline{m}}(t)$ is now replaced by a memoryless, nonlinearly distorted version. The function $2\Phi(x)-1$ is linear for small x, namely, $(2/\pi)^{\frac{1}{2}}\, x$; however, it saturates at $\pm 1$ as $x \to \pm\infty$.

In order to determine the covariance matrix of DIMUS output vector **Z**, consider the crosscorrelation of output variables $z_m$ and $z_{\ell}$ from equation (10):

$$E\{z_m\, z_{\ell}\} = \sum_{t,u=1}^{T} r_m(t)\, r_{\ell}(u) \sum_{k,j=1}^{K} E\{y_k(t)\, y_j(u)\} \ . \quad (17)$$

From equation (11), the required ensemble average follows as

$$E\{y_k(t)\, y_j(u)\} = E\{\mathrm{sgn}\{s_{\underline{m}}(t) + n_k(t)\}\, \mathrm{sgn}\{s_{\underline{m}}(u) + n_j(u)\}\} =$$

$$= \iint dx\, dy\, \mathrm{sgn}\{s_{\underline{m}}(t) + x\}\, \mathrm{sgn}\{s_{\underline{m}}(u) + y\}\, p_{kj}(x,y,t-u) \ , \quad (18)$$

where the joint second-order stationary Gaussian noise PDF $p_{kj}$ is

$$p_{kj}(x,y,\tau) = \frac{1}{2\pi\sigma_k\sigma_j\left(1-\rho_{kj}^2(\tau)\right)^{\frac{1}{2}}} \exp\left(-\frac{\left(\frac{x}{\sigma_k}\right)^2+\left(\frac{y}{\sigma_j}\right)^2-2\rho_{kj}(\tau)\frac{xy}{\sigma_k\sigma_j}}{2\left(1-\rho_{kj}^2(\tau)\right)}\right). \quad (19)$$

Covariance coefficient $\rho_{kj}(\tau) = R_{kj}(\tau)/(\sigma_k\sigma_j)$. Evaluation of double integral (18) is conducted in appendix B, with infinite expansion (B-7) as the end result.

The covariance of interest is given by equation (B-8) as

$$\mathrm{Cov}\{\mathbf{y}_k(t),\mathbf{y}_j(u)\} = C\left(\rho_{kj}(t-u), s_{\underline{m}}(t)/\sigma_k, s_{\underline{m}}(u)/\sigma_j\right) \quad (20)$$

for $1 \le k,j \le K$, where function

$$C(\rho,x,y) \equiv \frac{2}{\pi}\exp\left(-\frac{x^2+y^2}{2}\right)\sum_{n=0}^{\infty}\frac{\rho^{n+1}}{(n+1)!}\,He_n(x)\,He_n(y)\,. \quad (21)$$

A BASIC program for this form of function $C(\rho,x,y)$ is provided in appendix B. Additional forms for covariance $\mathrm{Cov}\{\mathbf{y}_k(t),\mathbf{y}_j(u)\}$, as well as for the limiting cases of $\rho = \pm 1$, are also presented in this appendix.

The covariance of DIMUS outputs $\mathbf{z}_m$ and $\mathbf{z}_\lambda$ now follows from equations (17) and (20) as

$$\mathrm{Cov}\{\mathbf{z}_m,\mathbf{z}_\lambda\} = \sum_{t,u=1}^{T} r_m(t)\; r_\lambda(u) \sum_{k,j=1}^{K} \mathrm{Cov}\{\mathbf{y}_k(t),\mathbf{y}_j(u)\} =$$

$$= \sum_{t,u=1}^{T} r_m(t)\; r_\lambda(u) \sum_{k,j=1}^{K} C\left(\rho_{kj}(t-u), s_{\underline{m}}(t)/\sigma_k, s_{\underline{m}}(u)/\sigma_j\right) \quad (22)$$

for $1 \le m,\lambda \le M$.

Equations (15) and (22) furnish the mean and covariance information required to statistically characterize the DIMUS output random variables in figure 2, at least for large KT. However, when that is true, equation (22), which requires that $(KT)^2$ terms be calculated and summed for each pair $m,\ell$, will entail a considerable amount of calculations.

If the expression for function C in equation (21) is substituted into equation (22), and if all the noise standard deviations $\sigma_k$ are equal to $\sigma$, the covariance can be expressed as

$$\mathrm{Cov}\{z_m, z_\ell\} = \frac{2}{\pi} \sum_{n=0}^{\infty} \frac{1}{(n+1)!} \sum_{t,u=1}^{T} r_m(t)\, r_\ell(u) \exp\left(-\frac{a^2(t) + b^2(u)}{2}\right) \times$$

$$\times\, He_n(a(t))\, He_n(b(u))\, F_n(t-u) \quad \text{for } 1 \le m,\ell \le M\ , \qquad (23)$$

where

$$a(t) \equiv s_{\underline{m}}(t)/\sigma\ , \quad b(u) \equiv s_{\underline{m}}(u)/\sigma\ , \quad F_n(\tau) \equiv \sum_{k,j=1}^{K} \rho_{kj}^{n+1}(\tau)\ . \qquad (24)$$

If the double sum on k,j in equation (24) can be evaluated in closed form (for some noise spectra), expansion (23) may afford a quicker method of evaluation than equation (22). Recursions on the Hermite functions in expansion (23) could still be used to advantage; however, most likely, there will not be a recursion for sequence $\{F_n(\tau)\}$.

## EXECUTION TIME

Some sample execution times for the evaluation of the covariance in equation (22) using a Hewlett-Packard 9000 computer are listed in table 1. These times are for a single value of the $m,\lambda$ pair, and it is presumed that the $r_m(t)$, $r_\lambda(u)$, $\rho_{kj}(t-u)$, and $s_{\underline{m}}(t)/\sigma_k$ arrays have already been computed and filled in. On the other hand, no use of any inherent symmetries have been taken advantage of, which would have reduced the execution time; rather, equation (22) was evaluated directly in the form given.

**Table 1. Sample Execution Times**

| K | T | Time (sec) |
|---|---|---|
| 10 | 12 | 41.1 |
| 10 | 24 | 166.7 |
| 20 | 24 | 656.9 |

For a CRAY computer, these times would be reduced by almost a factor of 1000, requiring about 1 second for K = 20, T = 24. It can be observed that the execution times behave proportionally to $K^2$ and $T^2$. Thus, K and/or T could be significantly increased for a comparative study of linear and DIMUS beamformers. However, the earlier case of K = 256, T = 4096 (reference 1) appears to require excessive computation time, even for a CRAY computer.

The execution time is heavily dependent on the sizes of the covariance coefficients $\{\rho_{kj}(t-u)\}$ encountered in equation (22), as well as on the error tolerance (Tol) used in the program in

appendix B. The timing results above are for a uniform distribution of $\rho$ values over $\pm 0.8$ and Tol = 1.E-8. If the $\rho$ values are uniformly distributed over $\pm 0.5$ and Tol is increased to 1.E-6, the execution time is reduced by half.

## APPENDIX A — GENERATION OF CORRELATED RANDOM VARIABLES WITH SPECIFIED MEAN AND COVARIANCE MATRICES

Let $\mathbf{U} = [u_1,\ldots,u_M]^T$ be a set of M zero-mean, unit-variance uncorrelated random variables. Form the linear transformation

$$\mathbf{V} = \underline{A}\,\mathbf{U} + \underline{B} \ , \tag{A-1}$$

where matrix $\underline{A}$ is MxM and matrix $\underline{B}$ is Mx1. Then, random Mx1 vector $\mathbf{V}$ has mean and covariance matrices given by

$$E\{\mathbf{V}\} = \overline{\mathbf{V}} = \underline{B} \ ,$$

$$\mathrm{Cov}\{\mathbf{V}\} = \underline{A}\,\overline{\mathbf{U}\,\mathbf{U}^T}\,\underline{A}^T = \underline{A}\,\underline{A}^T \ , \tag{A-2}$$

respectively, where an overbar denotes an ensemble average.

If the mean and covariance matrices of vector $\mathbf{V}$ are specified as $U_v$ and $C_v$, respectively, the two matrices in equation (A-1) must be taken according to $\underline{B} = U_v$ and $\underline{A}\,\underline{A}^T = C_v$. That is, $\underline{A}$ is a square root matrix of $C_v$.

If, in addition, all the random variables $\{u_m\}$ in equation (A-1) are Gaussian, then random variables $\mathbf{V}$ are joint Gaussian, because equation (A-1) is a linear transformation. In this case, equation (A-2) yields a complete statistical description of random variables $\mathbf{V}$.

## APPENDIX B — EVALUATION OF DOUBLE INTEGRAL (18)

The desired crosscorrelation is obtained by substituting joint PDF (19) into equation (18). Letting

$$a = s_{\underline{m}}(t)/\sigma_k \ , \ b = s_{\underline{m}}(u)/\sigma_j \ , \ x' = x/\sigma_k \ , \ y' = y/\sigma_j \ , \qquad \text{(B-1)}$$

and then replacing x′ and y′ by x and y, respectively, for notational simplicity results in

$$E\{y_k(t)\ y_j(u)\} = \iint dx\ dy\ \mathrm{sgn}\{a + x\}\ \mathrm{sgn}\{b + y\} \times$$

$$\times \frac{1}{2\pi(1 - \rho^2)^{\frac{1}{2}}} \exp\left(- \frac{x^2 + y^2 - 2\rho xy}{2(1 - \rho^2)}\right) \ ; \quad \rho \equiv \rho_{kj}(t-u) \ . \qquad \text{(B-2)}$$

At this point, the following expansion (reference 2, chapter 26) is employed, which separates the dependence on x,y,ρ:

$$\frac{1}{(1 - \rho^2)^{\frac{1}{2}}} \exp\left(- \frac{x^2+y^2-2\rho xy}{2(1 - \rho^2)}\right) = \exp\left(- \frac{x^2+y^2}{2}\right) \sum_{n=0}^{\infty} \frac{\rho^n}{n!}\, He_n(x)\, He_n(y) \ . \qquad \text{(B-3)}$$

Substitution in double integral (B-2) yields

$$E\{y_k(t)\ y_j(u)\} = \sum_{n=0}^{\infty} \frac{\rho^n}{n!}\, E_n(a)\, E_n(b) \ , \qquad \text{(B-4)}$$

where

$$E_n(a) \equiv \int dx\ \mathrm{sgn}\{a + x\}\ (2\pi)^{-\frac{1}{2}} \exp(-x^2/2)\, He_n(x) \ . \qquad \text{(B-5)}$$

There follows, upon use of equation (14), $E_0(a) = 2\Phi(a)-1$. By use of reference 2, equation 22.11.8, the remaining functions are

$$E_n(a) = \left(\frac{2}{\pi}\right)^{\frac{1}{2}} \exp(-a^2/2)\ (-1)^{n-1}\, He_{n-1}(a) \quad \text{for } n \geq 1 \ . \qquad \text{(B-6)}$$

Collecting these results, equation (B-4) becomes

$$E\{y_k(t)\ y_j(u)\} = [2\Phi(a)-1]\ [2\Phi(b)-1] +$$

$$+ \frac{2}{\pi} \exp\left(-\frac{a^2 + b^2}{2}\right) \sum_{n=1}^{\infty} \frac{\rho^n}{n!} He_{n-1}(a)\ He_{n-1}(b)\ . \qquad (B-7)$$

(This expansion is essentially the same as reference 2, equation 26.3.29.) When the identity of a and b in equation (B-1) is recalled, the leading term in equation (B-7) is recognized, by reference to equation (13), as the product of the mean values of $y_k(t)$ and $y_j(u)$. Then, shifting integer n by 1, the covariance follows from equation (B-7) as

$$Cov\{y_k(t), y_j(u)\} = C\left(\rho_{kj}(t-u), s_{\underline{m}}(t)/\sigma_k, s_{\underline{m}}(u)/\sigma_j\right) \qquad (B-8)$$

for $1 \le k,j \le K$, where function

$$C(\rho,x,y) \equiv \frac{2}{\pi} \exp\left(-\frac{x^2 + y^2}{2}\right) \sum_{n=0}^{\infty} \frac{\rho^{n+1}}{(n+1)!} He_n(x)\ He_n(y)\ . \qquad (B-9)$$

The Hermite polynomial $He_n(x)$ grows approximately as $(n!)^{\frac{1}{2}}$ for increasing n; see reference 3, equation 8.22.8. Therefore, direct recursions on each of the three multiplicative terms in the sum of equation (B-9) will likely encounter underflow and overflow before sufficient convergence is achieved. In order to alleviate these effects, a factor $\Gamma(n/2+1)\ 2^{n/2}$ was inserted in the denominators of both Hermite functions, and the square of that factor was applied to (multiplied by) the first term. This factor increases in a similar fashion to $(n!)^{\frac{1}{2}}$ for large n, but can be evaluated by a recursive procedure without square roots.

With the definitions

$$E = \exp(-(x^2+y^2)/4) \ , \quad A_n = \frac{(2\rho)^{n+1}\ \Gamma^2(n/2+1)}{\pi\ (n+1)!} \ ,$$

$$B_n = \frac{\Gamma((n+1)/2)}{\sqrt{2}\ \Gamma(n/2+1)} \ , \quad R_n^z = \frac{He_n(z)\ E}{\Gamma(n/2+1)\ 2^{n/2}} \ , \ z = x \text{ or } y \ , \qquad (B\text{-}10)$$

expansion (B-9) can be expressed as

$$C(\rho,x,y) = \sum_{n=0}^{\infty} A_n\ R_n^x\ R_n^y \ , \qquad (B\text{-}11)$$

with starting values and recursions (for $n \geq 2$)

$$A_0 = \frac{2}{\pi}\rho \ , \quad A_1 = \frac{1}{2}\rho^2 \ , \quad A_n = A_{n-2}\ \rho^2\ \frac{n}{n+1} \ ;$$

$$B_0 = \left(\frac{\pi}{2}\right)^{\frac{1}{2}} \ , \quad B_1 = \left(\frac{2}{\pi}\right)^{\frac{1}{2}} \ , \quad B_n = B_{n-2}\ \frac{n-1}{n} \ ;$$

$$R_0^z = E \ , \quad R_1^z = E\ z\ \left(\frac{2}{\pi}\right)^{\frac{1}{2}} \ , \quad R_n^z = R_{n-1}^z\ z\ B_n - R_{n-2}^z\ \frac{n-1}{n} \ . \qquad (B\text{-}12)$$

The last recursion was derived from reference 2, equation 22.7.14, and definition (B-10) above.

An efficient BASIC program for $C(\rho,x,y)$ that incorporates these modified functions is listed in figure B-1; it employs only the one exponential E in equation (B-10). The tolerance (Tol) and maximum number of terms utilized (Num) are under the user's control. The tolerance requirement is imposed on the sum of the magnitudes of the latest two terms in the series, relative to the current sum at that point.

```
 10   DEF FNC(P,X,Y)
 20   T=X*X+Y*Y
 30   IF (T>200.) THEN RETURN 0.
 40   Tol=1.E-8        !  SPECIFIED RELATIVE ERROR
 50   Num=1000         !  MAXIMUM NUMBER OF TERMS
 60   E=EXP(-.25*T)
 70   P2=P*P
 80   A2=.63661977236758134*P
 90   A1=.5*P2
100   B2=1.2533141373155003
110   B1=1./B2
120   R2x=E
130   R1x=E*X*B1
140   R2y=E
150   R1y=E*Y*B1
160   F1=.5
170   T1=1.E100
180   S=A2*R2x*R2y*(1.+.5*P*X*Y)
190   FOR N=2 TO Num
200   F=N/(N+1)
210   A=A2*P2*F
220   B=B2*F1
230   Rx=R1x*X*B-R2x*F1
240   Ry=R1y*Y*B-R2y*F1
250   T=A*Rx*Ry
260   S=S+T
270   IF (T1+ABS(T))<=(Tol*ABS(S)) THEN 410
280   A2=A1
290   A1=A
300   B2=B1
310   B1=B
320   R2x=R1x
330   R1x=Rx
340   R2y=R1y
350   R1y=Ry
360   F1=F
370   T1=ABS(T)
380   NEXT N
390   PRINT "TOLERANCE NOT SATISFIED"
400   PAUSE
410   RETURN S
420   FNEND
```

**Figure B-1. Program for Function C(ρ,x,y) in Equation (B-9)**

With the aid of equations (13), (14), (B-1), and (B-2), the covariance can be manipulated into the single integral form

$$\mathrm{Cov}\{y_k(t), y_j(u)\} = 4 \int_a^\infty dx\ \phi(x) \left[\Phi(b) - \Phi\left(\frac{b - \rho x}{(1 - \rho^2)^{1/2}}\right)\right] \ ; \quad \text{(B-13)}$$

the alternative form with a and b switched is also allowed.

For $\rho = \pm 1$, none of the above forms are directly usable. Instead, a direct evaluation of the covariance in these two special cases yields

$$\mathrm{Cov} = \begin{cases} 4\ \Phi(a)\ \Phi(-b) & \text{for } a < b \\ 4\ \Phi(-a)\ \Phi(b) & \text{for } a > b \end{cases} \quad \text{for } \rho = +1 \qquad \text{(B-14)}$$

and

$$\mathrm{Cov} = \begin{cases} -4\ \Phi(a)\ \Phi(b) & \text{for } a+b < 0 \\ -4\ \Phi(-a)\ \Phi(-b) & \text{for } a+b > 0 \end{cases} \quad \text{for } \rho = -1\ . \qquad \text{(B-15)}$$

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